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# Vector-multiplet effective action in the non-anticommutative charged hypermultiplet model

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## Abstract

We investigate the quantum aspects of a charged hypermultiplet in deformed $\mathcal{N}=(1,1)$ superspace with singlet non-anticommutative deformation of supersymmetry. This model is a "star" modification of the hypermultiplet interacting with a background Abelian vector superfield. We prove that this model is renormalizable in the sense that the divergent part of the effective action is proportional to the $\mathcal{N}=(1,0)$ non-anticommutative super Yang-Mills action. We also calculate the finite part of the low-energy effective action depending on the vector multiplet, which corresponds to the (anti)holomorphic potential. The holomorphic piece is just deformed to the star-generalization of the standard holomorphic potential, while the antiholomorphic piece is not. We also reveal the component structure and find the deformation of the mass and the central charge.

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# 1 Introduction and Summary

In this paper we study the quantum aspects of $\mathcal{N}$=$(1,0)$ non-anticommutative theories with singlet deformation of supersymmetry. To provide the motivation of this work let us briefly summarize the most important achievements and problems concerning non-anticommutative theories with $\mathcal{N}$=$(1/2,0)$ and $\mathcal{N}$=$(1,0)$ supersymmetry.

The interest in $\mathcal{N}$=$(1/2,0)$ non-anticommutative deformations of supersymmetry originated with the papers [1, 2, 3], where these deformations were derived from superstring theory on a constant graviphoton background. As a rule, such deformations break supersymmetry only in the chiral sector of superspace, which is possible in Euclidean superspace. The key feature of non-anticommutative deformations on the quantum level is the preservation of renormalizability, which was established for $\mathcal{N}$=$(1/2,0)$ Wess-Zumino [4] and super Yang-Mills (SYM) [5, 6, 7] models. This result is very non-trivial since the non-anticommutative deformations involve a parameter with negative mass dimension which plays the role of a new coupling constant. Since such theories appear to be renormalizable, their quantum dynamics should be explored. Indeed, the low-energy effective action of such models was considered in [8], where the corrections due to the non-anticommutative deformation were calculated. These results provide a promising new method of partial supersymmetry breaking which preserves renormalizability. New remarkable quantum features suggest interesting physical applications (see, e.g., [2] for modifications of the glueball superpotential and the expectation value of the glueball field in $\mathcal{N}$=$(1/2,0)$ supersymmetric field theories).

These surprises of $\mathcal{N}$=$(1/2,0)$ theories motivated analogous investigations of deformed *extended* supersymmetric theories. In this case we distinguish several types of non-anticommutative deformations. The simplest one depends on a single scalar parameter $I$ which appears in the anticommutator of the chiral $\mathcal{N}$= $(1,1)$ Grassmann coordinates,

$$\{\theta^\alpha_i, \theta^\beta_j\}_\star = 2I\varepsilon^{\alpha\beta}\varepsilon_{ij}. \tag{1.1}$$

Such a deformation was introduced in [9] and was named a chiral singlet deformation; the corresponding field theories are referred to as $\mathcal{N}$=$(1,0)$ non-anticommutative. Although more general types of deformations of extended supersymmetry have been considered in [10, 11], here we shall restrict ourselves to the singlet deformation (1.1), since this case is most elaborated now on the classical level [12]-[15] and its stringy origins have been established [12].

The quantum aspects of $\mathcal{N}$=$(1,0)$ non-anticommutative theories are more involved. In [12, 13, 14] it was shown that the $\mathcal{N}$=$(1,0)$ SYM and hypermultiplet models acquire a number of new classical interaction terms even in the Abelian case. In principle, such terms can spoil renormalizability. In our recent paper [16] we addressed this problem in two such $\mathcal{N}$=$(1,0)$ theories, namely the Abelian SYM model and the *neutral* hypermultiplet interacting with an Abelian gauge superfield. By computing all divergent contributions to the effective action we proved that both these models remain renormalizable. Note that these theories are deformations of free ones, and thus all interaction terms vanish in the undeformed limit $I \to 0$. A physically more important example is the *charged* hypermultiplet model, which – already prior to the deformation – features the interaction of a hypermultiplet with a background Abelian vector superfield. In particular, the low-energy effective action of this theory is goverend by the so-called holomorphic potential, which plays a significant role in the Seiberg-Witten theory [17]. Therefore, in the present work we study the low-energy effective action and renormalizability for the $\mathcal{N}{=}(1,0)$ non-anticommutative charged hypermultiplet model.

Theories with extended supersymmetry are most naturally described within the harmonic superspace approach [18, 19]. Hence, we will consider the $\mathcal{N}{=}(1,0)$ non-anticommutative charged hypermultiplet in the harmonic superspace that was studied on the classical level in [13, 14]. We are interested in the non-anticommutative deformation of the holomorphic effective action that was discussed in [20, 21] using the harmonic superspace approach. Here we generalize the results of these works to the non-anticommutatively deformed hypermultiplet theory and compute the leading contributions to the effective action.

One of our main results is the proof of one-loop renormalizability of the deformed charged hypermultiplet system. It supports the idea that the non-anticommutative deformations in general do not spoil the renormalizability of supersymmetric theories. Next, we find the leading contributions to the (anti)holomorphic effective action including non-anticommutative corrections. We observe that the holomorphic and antiholomorphic pieces are deformed differently: the holomorphic piece is nothing but the star-generalization of the standard holomorphic potential while the antiholomorphic piece is not. We study also the component structure of the deformed effective action and derive the corrections to the standard terms in the (anti)holomorphic potential for the bosonic component fields. The deformation of the mass and the central charge due to non-anticommutativity are found as well.

The paper is organized as follows. In Sect. 2 we review the basic aspects of the undeformed charged hypermultiplet theory for later comparison with the deformed ones. Sect. 3 summarizes the most essential points about the $\mathcal{N}{=}(1,0)$ non-anticommutative charged hypermultiplet on the classical level. In Sect. 4 the leading contributions to the low-energy effective action are computed, culminating in the renormalizability proof. In Sect. 5 we derive the component structure of the (anti)holomorphic effective action found in the previous section and analyze the new terms which appear due to the deformation. Sect. 6 contains our comments on the non-anticommutative deformation of mass and central charge in the charged hypermultiplet model. In the Conclusions we discuss the obtained results and point out some tempting unsolved problems. An Appendix collects some properties of a special function which encodes the deformation of the holomorphic potential.

# 2 The undeformed theory

In this section we review briefly the known results concerning the model of charged hypermultiplet and its effective action.

The $\mathcal{N}=2$ supersymmetric models are most naturally described within the $\mathcal{N}=2$ harmonic superspace approach [18, 19]. In particular, the classical action of charged hypermultiplet interacting with the Abelian vector superfield is given by

$$S=\int d\zeta^{(-4)}du\,\breve{q}^{+}(D^{++}+V^{++})q^{+}. \tag{2.1}$$

Here $q^+$ and its conjugate $\breve{q}^+$ are complex analytic superfields which describe the hypermultiplet, while $V^{++}$ is a real analytic superfield which corresponds to the vector multiplet. The integration in (2.1) is performed over $\mathcal{N}=2$ analytic superspace with the measure $d\zeta^{(-4)}du$. The action (2.1) is invariant under the following (Abelian) gauge transformations of superfields

$$\delta q^{+}=\lambda q^{+},\qquad \delta\breve{q}^{+}=-\breve{q}^{+}\lambda,\qquad \delta V^{++}=-D^{++}\lambda, \tag{2.2}$$

where $\lambda$ is a real analytic superfield.

The general structure of the low-energy effective action in the model (2.1) is given by (see, e.g., [20, 21])

$$\Gamma=\int d^4xd^4\theta\,\mathcal{F}(W)+\int d^4xd^4\bar{\theta}\,\bar{\mathcal{F}}(\bar{W})+\int d^4xd^8\theta\,\mathcal{H}(W,\bar{W}), \tag{2.3}$$

where $\mathcal{F}$ is holomorphic potential, $\bar{\mathcal{F}}$ is antiholomorphic potential, $\mathcal{H}$ is non-holomorphic potential. The strength superfields $W$, $\bar{W}$ are gauge invariant objects which are expressed through the gauge prepotential as follows

$$\bar{W}=-\frac{1}{4}D^{+\alpha}D^{+}_{\alpha}V^{--},\qquad W=-\frac{1}{4}\bar{D}^{+}_{\dot{\alpha}}\bar{D}^{+\dot{\alpha}}V^{--}. \tag{2.4}$$

where

$$V^{--}(z,u)=\int du_1\frac{V^{++}(z,u_1)}{(u^+u_1^+)^2} \tag{2.5}$$

is a solution of the zero-curvature equation

$$D^{++}V^{--}-D^{--}V^{++}=0. \tag{2.6}$$

Note that in the Abelian case the superfields $W$ and $\bar{W}$ are chiral and antichiral respectively

$$D^{\pm}_{\alpha}\bar{W}=0,\qquad \bar{D}^{\pm}_{\dot{\alpha}}W=0. \tag{2.7}$$

Therefore the functions $\mathcal{F}(W)$ and $\bar{\mathcal{F}}(\bar{W})$, which are related to each other by complex conjugation, are integrated over the chiral and antichiral superspaces respectively.

The perturbative low-energy effective action in the model (2.1) is studied now in details, (see, e.g., [20]-[25]) where both holomorphic and non-holomorphic contributions have been found. In particular, the holomorphic part of the effective action which is leading in the low-energy approximation is given by

$$\mathcal{F}(W) = -\frac{1}{32\pi^2} W^2 \ln \frac{W}{\Lambda}, \tag{2.8}$$

where $\Lambda$ is some scale.

The strength superfields $W$, $\bar{W}$ have the following component structure in the bosonic sector

$$W = \phi + (\theta^+ \sigma_{mn} \theta^-) F_{mn} + \ldots, \qquad \bar{W} = \bar{\phi} + (\bar{\theta}^+ \tilde{\sigma}_{mn} \bar{\theta}^-) F_{mn} + \ldots, \tag{2.9}$$

where $\phi$, $\bar{\phi}$ are scalar fields, $F_{mn} = \partial_m A_n - \partial_n A_m$ is the Maxwell strength and dots stand for the terms with spinors $\Psi^i_\alpha$, $\bar{\Psi}_{i\dot{\alpha}}$ and auxiliary fields $\mathcal{D}^{kl}$ as well as the terms with spatial derivatives. The component structure of holomorphic effective action can be most easily derived in the following approximation

$$\begin{gathered} \phi = const, \quad \bar{\phi} = const, \quad F_{mn} = const, \\ \Psi^i_\alpha = \bar{\Psi}_{i\dot{\alpha}} = \mathcal{D}^{kl} = 0. \end{gathered} \tag{2.10}$$

Substituting the strength superfields (2.9) into (2.3), one gets the component structure of the holomorphic part of the effective action

$$\Gamma_{hol} = \int d^4x d^4\theta\, \mathcal{F}(W) = -\frac{1}{32\pi^2} \int d^4x (F_{mn} F_{mn} + F_{mn} \tilde{F}_{mn}) \left( \ln \frac{\phi}{\Lambda} + \frac{3}{2} \right) + \ldots, \tag{2.11}$$

where $\tilde{F}_{mn} = \frac{1}{2} \varepsilon_{mnrs} F_{rs}$ and dots stand for the higher terms which are not essential in the approximation (2.10). The constant 3/2 in (2.11) can be removed by the shift of the parameter $\Lambda$, however it will be important when we will consider the deformation of (anti)holomorphic effective action due to non-anticommutativity. The antiholomorphic part of effective action is given by the complex conjugation of the action (2.11).

# 3 Non-anticommutative charged hypermultiplet model

The chiral singlet deformation of $\mathcal{N} = (1,1)$ superspace was introduced in [9] and the corresponding field models were studied in [12]-[16]. Such a deformation is effectively taken into account by the star product operator

$$\star = \exp \left[ -I \varepsilon^{\alpha\beta} \varepsilon_{ij} \overleftarrow{Q}^i_\alpha \overrightarrow{Q}^j_\beta \right], \tag{3.1}$$

which should be placed everywhere instead of usual product of superfields in the classical actions. The constant $I$ here is a parameter of non-anticommutativity, $Q^i_\alpha$ are the supercharges. In particular, the non-anticommutative generalization of the action (2.1) is given by [13]

$$S = \int d\zeta^{(-4)} du\, \breve{q}^+ \star \nabla^{++} \star q^+, \tag{3.2}$$

where we use the notations

$$\nabla^{++} = D^{++} + V^{++}, \qquad \nabla^{--} = D^{--} + V^{--}. \tag{3.3}$$

This action is invariant under the following gauge transformations

$$\delta\breve{q}^{+} = -\breve{q}^{+} \star \lambda, \qquad \delta q^{+} = \lambda \star q^{+}, \qquad \delta V^{++} = -D^{++}\lambda - [V^{++}, \lambda]_{\star}. \tag{3.4}$$

which are the non-anticommutative generalizations of the usual ones (2.2).

In the deformed case the strength superfields $W$, $\bar{W}$ are defined by the standard equations (2.4), but the superfield $V^{--}$ is now given by a series

$$V^{--}(z,u) = \sum_{n=1}^{\infty}(-1)^n \int du_1 \ldots du_n \frac{V^{++}(z,u_1) \star V^{++}(z,u_2) \star \ldots \star V^{++}(z,u_n)}{(u^+u_1^+)(u_1^+u_2^+)\ldots(u_n^+u^+)}, \tag{3.5}$$

which solves the star-deformed zero-curvature equation [12]

$$D^{++}V^{--} - D^{--}V^{++} + [V^{++}, V^{--}]_{\star} = 0. \tag{3.6}$$

Let us introduce the "bridge" superfield $\Omega(z,u)$ as a general $\mathcal{N} = (1,1)$ superfield which relates the covariant harmonic derivatives $\nabla^{\pm\pm}$ with the plain ones $D^{\pm\pm}$:

$$\nabla^{++} = e_{\star}^{\Omega} \star D^{++} e_{\star}^{-\Omega}, \qquad \nabla^{--} = e_{\star}^{\Omega} \star D^{--} e_{\star}^{-\Omega}. \tag{3.7}$$

The bridge superfield was originally introduced in [18] for the undeformed $\mathcal{N} = 2$ SYM theory as an operator relating the $\mathcal{N} = 2$ superfields in the $\tau$- and $\lambda$-frames. Using the bridge superfield $\Omega$ one can alternatively rewrite the equation (3.5) in the following two equivalent forms

$$V^{--}(z,u) = \int du' \frac{e_{\star}^{\Omega(z,u)} \star e_{\star}^{-\Omega(z,u')} \star V^{++}(z,u')}{(u^+u'^+)^2} = \int du' \frac{V^{++}(z,u') \star e_{\star}^{\Omega(z,u')} \star e_{\star}^{-\Omega(z,u)}}{(u^+u'^+)^2}. \tag{3.8}$$

The expression (3.8) can be checked directly to satisfy the zero-curvature condition (3.6).

It is well known [19] that the *free* propagator in the hypermultiplet model (2.1) is given by the following expression [1]

$$G_0^{(1,1)}(1|2) = -\frac{1}{\Box}(D_1^+)^4(D_2^+)^4 \frac{\delta^{12}(z_1 - z_2)}{(u_1^+u_2^+)^3} \tag{3.9}$$

which solves the equation $D^{++}G_0^{(1,1)}(1|2) = \delta_A^{(3,1)}(1|2)$, where $\delta_A^{(3,1)}(1|2)$ is the analytic delta-function. Let us define now the *full* propagator in the model (3.2) as a distribution satisfying the equation

$$\nabla^{++} \star G^{(1,1)}(1|2) = \delta_A^{(3,1)}(1|2). \tag{3.10}$$

[1]Note that we write here (and further) the box operator $\Box$ assuming that it is nothing but the Laplacian operator rather than a d'Alambertian one since we deal with the Euclidian space.

The solution of (3.10) can formally be written as

$$G^{(1,1)}(1|2) = -\frac{1}{\hat{\Box}_\star} \star (D_1^+)^4 (D_2^+)^4 \left\{ e_\star^{\Omega(1)} \star e_\star^{-\Omega(2)} \star \frac{\delta^{12}(z_1 - z_2)}{(u_1^+ u_2^+)^3} \right\}, \tag{3.11}$$

where $\hat{\Box}_\star$ is a covariant box operator

$$\hat{\Box}_\star = -\frac{1}{2}(D^+)^4 \nabla^{--} \star \nabla^{--}. \tag{3.12}$$

Clearly, it moves an analytic superfield to another analytic one. The operator (3.12), acting on the analytic superfield, can be represented in the form

$$\begin{aligned} \hat{\Box}_\star \;=\; & \nabla^m \star \nabla_m - \frac{1}{2}(\nabla^{+\alpha} \star W) \star \nabla^-_\alpha - \frac{1}{2}(\bar{\nabla}^+_{\dot\alpha} \star \bar{W}) \star \bar{\nabla}^{-\dot\alpha} + \frac{1}{4}(\nabla^{+\alpha} \star \nabla^+_\alpha \star W) \star \nabla^{--} \\ & - \frac{1}{8}[\nabla^{+\alpha}, \nabla^-_\alpha]_\star \star W - \frac{1}{2}\{W, \bar{W}\}_\star. \end{aligned} \tag{3.13}$$

Here $\nabla^\pm_\alpha = D^\pm_\alpha + V^\pm_\alpha$, $\bar{\nabla}^\pm_{\dot\alpha} = \bar{D}^\pm_{\dot\alpha} + \bar{V}^\pm_{\dot\alpha}$ are covariant spinor derivatives. Note that the expression (3.13) has a similar form as in the undeformed theory [23] with the simple star-modification of multiplication of superfields. This result is not surprising since eq. (3.13) is derived from (3.12) only by using the (anti)commutation relations between spinor derivatives which have the same form as in the undeformed theory.

# 4 Computation of low-energy effective action

## 4.1 General structure of low-energy effective action

The model (3.2) is non-anticommutative deformation of the model (2.1). One can think naively that the effective potentials $\mathcal{F}$, $\bar{\mathcal{F}}$, $\mathcal{H}$ in the effective action (2.3) will get the analogous deformation

$$\mathcal{F}(W) \longrightarrow \mathcal{F}_\star(W), \qquad \bar{\mathcal{F}}(\bar{W}) \longrightarrow \bar{\mathcal{F}}_\star(\bar{W}), \qquad \mathcal{H}(W, \bar{W}) \longrightarrow \mathcal{H}_\star(W, \bar{W}). \tag{4.1}$$

However, we will show that this assertion is not true for the antiholomorphic potential in the sense that no any action can be constructed with a function $\bar{\mathcal{F}}_\star(\bar{W})$ integrated over the antichiral superspace. Indeed, the strength superfields $W$, $\bar{W}$ are not (anti)chiral, but *covariantly* (anti)chiral

$$D^+_\alpha \bar{W} = \nabla^-_\alpha \star \bar{W} = 0, \qquad \bar{D}^+_{\dot\alpha} W = \bar{\nabla}^-_{\dot\alpha} \star W = 0. \tag{4.2}$$

Therefore the expression $\int d^4x d^4\bar\theta \bar{\mathcal{F}}_\star(\bar{W})$ depends on $\theta$ variables

$$D^-_\alpha \int d^4x d^4\bar\theta\, \bar{\mathcal{F}}_\star(\bar{W}) = \int d^4x d^4\bar\theta\, [\bar{\mathcal{F}}_\star(\bar{W}), V^-_\alpha]_\star \neq 0. \tag{4.3}$$

The rhs of (4.3) does not vanish since the star-product is not cyclic under $d^4xd^4\bar\theta$ integration. Moreover, the expression $\int d^4xd^4\bar\theta\bar{\mathcal{F}}_\star(\bar W)$ violates the gauge invariance. Indeed, the strength superfields transform covariantly under the gauge transformations (3.4) of gauge superfield

$$\delta W = [\lambda, W]_\star, \qquad \delta\bar W = [\lambda, \bar W]_\star. \tag{4.4}$$

Since the superfields $\lambda$ and $\bar W$ are not antichiral, we have

$$\delta \int d^4xd^4\bar\theta\, \bar{\mathcal{F}}_\star(\bar W) = \int d^4xd^4\bar\theta\, [\lambda, \bar{\mathcal{F}}_\star(\bar W)]_\star \neq 0. \tag{4.5}$$

The similar remarks on the gauge invariance in holomorphic and antiholomorphic parts of classical action in $\mathcal{N} = (1/2, 0)$ gauge theory are given in [7]. Note also that there is no such a problem with the holomorphic potential $\mathcal{F}_\star(W)$, as it is pointed out in [12] for the case of classical $\mathcal{N} = (1, 0)$ SYM action.

Taking into account these remarks we propose that the general form of the low-energy effective action in the hypermultiplet model (3.2) is given by

$$\Gamma = \int d^4xd^4\theta\, \mathcal{F}_\star(W) + \int d^4xd^8\theta\, \mathcal{H}_\star(V^{++}, V^{--}, W, \bar W). \tag{4.6}$$

Here we assume that the possible terms in the low-energy effective action which correspond to the antiholomorphic potential in the limit $I \to 0$ is included into the function $\mathcal{H}_\star(V^{++}, V^{--}, W, \bar W)$ integrated in full superspace. The direct computations will specify these functions $\mathcal{F}_\star$ and $\mathcal{H}_\star$.

For the further considerations it will be more convenient to study the variation of effective action $\delta\Gamma$ rather then $\Gamma$ itself. In particular, given the holomorphic part of the action (4.6)

$$\Gamma_{hol} = \int d^4xd^4\theta\, \mathcal{F}_\star(W), \tag{4.7}$$

using the same steps as in the undeformed non-Abelian $\mathcal{N} = 2$ supergauge theory [25], one can write its variation either in the analytic superspace

$$\delta\Gamma_{hol} = \int d\zeta^{(-4)}du\, \delta V^{++} \star [-\frac{1}{4} D^{+\alpha} D^+_\alpha \mathcal{F}'_\star(W)], \tag{4.8}$$

or in the full superspace

$$\delta\Gamma_{hol} = \int d^{12}zdu\, \delta V^{++} \star V^{--} \star \frac{1}{W} \star \mathcal{F}'_\star(W). \tag{4.9}$$

## 4.2 One-loop effective action

The one-loop effective action in the model (3.2) is defined by the following formal expression [2]

$$\Gamma = \mathrm{Tr}\ln \frac{\delta^2 S}{\delta \breve{q}^+(1)\delta q^+(2)} = \mathrm{Tr}\ln(\nabla^{++}\star) = -\mathrm{Tr}\ln\, G^{(1,1)}(1|2), \tag{4.10}$$

where $G^{(1,1)}(1|2)$ is given by eq. (3.11). It is easy to find the variation of (4.10)

$$\delta\Gamma = \mathrm{Tr}[\delta V^{++}\star G^{(1,1)}] = \int d\zeta^{(-4)}du\, \delta V^{++}(1)\star G^{(1,1)}(1|2)|_{(1)=(2)}. \tag{4.11}$$

There is an important relation between full and free hypermultiplet propagators

$$G^{(1,1)}(1|3) = G_0^{(1,1)}(1|3) - \int d\zeta_2^{(-4)}du_2\, G_0^{(1,1)}(1|2)\star V^{++}(2)\star G^{(1,1)}(2|3) \tag{4.12}$$

which can be checked directly to satisfy (3.10). Substituting (4.12) into (4.11), we find

$$\delta\Gamma = -\int d\zeta_1^{(-4)}du_1 d\zeta_2^{(-4)}du_2\, \delta V^{++}(1)\star G_0^{(1,1)}(1|2)\star V^{++}(2)\star G^{(1,1)}(2|1). \tag{4.13}$$

Taking into account the exact form of the propagators (3.9,3.11), we rewrite eq. (4.13) as follows

$$\begin{aligned}\delta\Gamma \;=\; & -\int d\zeta_1^{(-4)}d\zeta_2^{(-4)}du_1du_2\, \delta V^{++}(1)\star \frac{1}{\Box}(D_1^+)^4(D_2^+)^4\frac{\delta^{12}(z_1-z_2)}{(u_1^+u_2^+)^3} \\ & \times V^{++}(2)\star\frac{1}{\hat{\Box}_{\star(2)}}\star(D_1^+)^4(D_2^+)^4\left\{e_\star^{\Omega(2)}\star e_\star^{-\Omega(1)}\frac{\delta^{12}(z_2-z_1)}{(u_2^+u_1^+)^3}\right\}.\end{aligned} \tag{4.14}$$

Now we take off the derivatives $(D_1^+)^4(D_2^+)^4$ from the first delta-function to restore the full superspace measure

$$\begin{aligned}\delta\Gamma \;=\; & -\int d^{12}z_1d^{12}z_2du_1du_2\, \delta V^{++}(1)\star \frac{1}{\Box}\frac{\delta^{12}(z_1-z_2)}{(u_1^+u_2^+)^3} \\ & \times V^{++}(2)\star\frac{1}{\hat{\Box}_{\star(2)}}\star(D_1^+)^4(D_2^+)^4\left\{e_\star^{\Omega(2)}\star e_\star^{-\Omega(1)}\star\frac{\delta^{12}(z_2-z_1)}{(u_2^+u_1^+)^3}\right\}.\end{aligned} \tag{4.15}$$

The equation (4.15) is a starting point for further calculations of different contributions to the effective action.

[2] Note that the one-loop effective action in the Euclidean space is given by $\Gamma = \mathrm{Tr}\ln S^{(2)}_{\phi\bar\phi}$ rather then the Minkowski space expression $\Gamma = i\mathrm{Tr}\ln S^{(2)}_{\phi\bar\phi}$. Here $S^{(2)}_{\phi\bar\phi}$ is the second mixed functional derivative of a classical action $S[\phi,\bar\phi]$.

### 4.3 Divergent part of effective action

To derive the divergent part of the effective action it is sufficient to consider the approximation

$$\frac{1}{\hat{\Box}_\star} \approx \frac{1}{\Box} \tag{4.16}$$

since all other terms in the operator $\hat{\Box}_\star$ result to higher powers of momenta in the denominator. Upon the condition (4.16), the variation of effective action (4.15) simplifies essentially

$$\begin{aligned}\delta\Gamma_{div} &= \int d^{12}z_1 d^{12}z_2 \frac{du_1 du_2}{(u_1^+u_2^+)^6}\delta V^{++}(1) \star \frac{1}{\Box}\delta^{12}(z_1-z_2) \\ &\times V^{++}(2) \star \frac{1}{\Box}(D_1^+)^4(D_2^+)^4 \left\{ e_\star^{\Omega(2)} \star e_\star^{-\Omega(1)} \star \delta^{12}(z_2-z_1) \right\}.\end{aligned} \tag{4.17}$$

We have to apply the identity

$$\delta^8(\theta_1-\theta_2)(D_1^+)^4(D_2^+)^4\delta^{12}(z_1-z_2) = (u_1^+u_2^+)^4\delta^{12}(z_1-z_2) \tag{4.18}$$

to shrink the integration over the Grassmann variables to a point. Note that all the derivatives $D^+$ in eq. (4.17) must hit the delta function, otherwise the result is zero since there are exactly eight such derivatives to apply (4.18). Moreover, the presence of star-product can not modify the relation (4.18). Calculating the divergent momentum integral

$$\left[ \int d^4k \frac{1}{k^2(p+k)^2} \right]_{\rm div} = \frac{\pi^2}{\varepsilon}, \qquad (\varepsilon \to 0) \tag{4.19}$$

and applying the equation (4.18), the integration over $d^{12}z_2$ in (4.17) can be performed resulting to

$$\delta\Gamma_{div} = \frac{1}{16\pi^2\varepsilon}\int d^{12}z du_1\, \delta V^{++}(z,u_1) \star \int du_2 \frac{V^{++}(z,u_2) \star e_\star^{\Omega(z,u_2)} \star e_\star^{-\Omega(z,u_1)}}{(u_1^+u_2^+)^2}. \tag{4.20}$$

Using the relation (3.8) we obtain finally

$$\delta\Gamma_{div} = \frac{1}{16\pi^2\varepsilon}\int d^{12}z du\, \delta V^{++} \star V^{--}. \tag{4.21}$$

The variation (4.21) can be easily integrated with the help of eq. (4.9)

$$\Gamma_{div} = \frac{1}{32\pi^2\varepsilon}\int d^4x d^4\theta\, W^2. \tag{4.22}$$

We see that the divergent part of effective action is proportional to the classical action in $\mathcal{N}=(1,0)$ SYM model. In this sense the non-anticommutative charged hypermultiplet model (3.2) is renormalizable.

## 4.4 Finite part of the effective action

Now we will derive the finite part of the effective action of deformed hypermultiplet model. We start with the expression (4.15) applying the following approximation

$$\frac{1}{\hat{\Box}_\star} \approx \frac{1}{\Box - \frac{1}{2}\{W,\bar{W}\}_\star}. \tag{4.23}$$

This means that we neglect all spatial and spinor covariant derivatives of strength superfields in the decomposition (3.13)

$$\partial_m W = \partial_m \bar{W} = 0, \quad \nabla^{+\alpha} \star W = 0, \quad \bar{\nabla}^+_{\dot{\alpha}} \star \bar{W} = 0. \tag{4.24}$$

Exactly such an approximation (4.24) is sufficient for deriving the (anti)holomorphic contributions. Therefore the effective action (4.15) can be rewritten as follows

$$\begin{aligned}\delta\Gamma &= \int d^{12}z_1 d^{12}z_2 \frac{du_1 du_2}{(u_1^+u_2^+)^6}\delta V^{++}(1) \star \frac{1}{\Box}\delta^{12}(z_1 - z_2) \\ &\times V^{++}(2) \star \frac{1}{\Box - \frac{1}{2}\{W,\bar{W}\}_\star}(D_1^+)^4(D_2^+)^4 \left\{e_\star^{\Omega(2)} \star e_\star^{-\Omega(1)} \star \delta^{12}(z_2 - z_1)\right\}. \end{aligned}\tag{4.25}$$

Once again, all the derivatives $D^+$ in the second line of (4.25) must hit the delta-function. Applying the identity (4.18) the integration over $d^8\theta_2$ is performed

$$\begin{aligned}\delta\Gamma &= \int d^{12}z_1 d^4x_2 \frac{du_1du_2}{(u_1^+u_2^+)^2}\delta V^{++}(x_1,\theta,u_1) \star V^{++}(x_2,\theta,u_2) \star e_\star^{\Omega(2)} \star e_\star^{-\Omega(1)} \\ &\star \frac{1}{\Box}\delta^4(x_1 - x_2)\frac{1}{\Box - \frac{1}{2}\{W,\bar{W}\}_\star}\delta^4(x_2 - x_1). \end{aligned}\tag{4.26}$$

In the momentum space the second line of (4.26) reads

$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\frac{1}{k^2 + \frac{1}{2}\{W,\bar{W}\}_\star} = -\frac{1}{16\pi^2}\ln_\star\left[\frac{\{W,\bar{W}\}_\star}{2\Lambda^2}\right] + (\text{divergent term}), \tag{4.27}$$

where $\Lambda$ is an arbitrary constant of dimension $+1$. Note that the integral (4.27) has the logarithmic divergence. We consider here only its finite part since the divergent contribution have been calculated above. As a result, the finite part of (4.26) is given by

$$\begin{aligned}\delta\Gamma &= -\frac{1}{16\pi^2}\int d^{12}z du_1\, \delta V^{++}(x,\theta,u_1) \star \int du_2 \frac{V^{++}(x,\theta,u_2) \star e_\star^{\Omega(x,\theta,u_2)} \star e_\star^{-\Omega(x,\theta,u_1)}}{(u_1^+u_2^+)^2} \\ &\star \ln_\star \frac{\{W,\bar{W}\}_\star}{2\Lambda^2}. \end{aligned}\tag{4.28}$$

Applying the identity (3.8), we conclude

$$\delta\Gamma = -\frac{1}{16\pi^2}\int d^{12}z du\, \delta V^{++} \star V^{--} \star \ln_\star \frac{\{W,\bar{W}\}_\star}{2\Lambda^2}. \tag{4.29}$$

The variation of the effective action (4.29) is one of the main results of the present work. It gives us the part of the low-energy effective action depending on the strength superfields without derivatives.

If the parameter of non-anticommutativity tends to zero, $I \to 0$, the equation (4.29) reduces to the holomorphic and antiholomorphic parts of the effective action

$$\delta\Gamma_{(I=0)} = -\frac{1}{16\pi^2}\int d^{12}z du\, \delta V^{++} V^{--} \left(\ln\frac{W}{\Lambda} + \ln\frac{\bar{W}}{\Lambda}\right). \quad (4.30)$$

The variation (4.30) exactly corresponds to the holomorphic potential (2.8) and its conjugate.

Note that when $I \neq 0$, the logarithm in (4.29) can not be represented in a form of a sum of holomorphic and antiholomorphic parts. Therefore, the variation of the effective action (4.29) is responsible for both holomorphic, antiholomorphic and non-holomorphic contributions to the effective action.

## 4.5 Holomorphic contribution

Let us single out purely holomorphic part from the expression (4.29). For this purpose we restrict the background strength superfield $\bar{W}$ to be constant

$$\bar{W} = \bar{\mathbf{W}} = const, \quad (4.31)$$

and the log function in (4.29) simplifies to

$$\ln_\star \frac{\{W, \bar{W}\}_\star}{2\Lambda^2} = \ln_\star \frac{W}{\Lambda} + \ln\frac{\bar{\mathbf{W}}}{\Lambda}. \quad (4.32)$$

The holomorphic part is now given by

$$\delta\Gamma_{hol} = -\frac{1}{16\pi^2}\int d^{12}z du\, \delta V^{++} \star V^{--} \star \ln_\star \frac{W}{\Lambda}. \quad (4.33)$$

According to the equation (4.9), the variation (4.33) can easily be integrated:

$$\Gamma_{hol} = -\frac{1}{32\pi^2}\int d^4x d^4\theta\, W \star W \star \ln_\star \frac{W}{\Lambda}. \quad (4.34)$$

As a result, we proved that the holomorphic part of effective action in the hypermultiplet model is nothing but a star-product generalization of a standard holomorphic potential (2.8).

## 4.6 Antiholomorphic contribution

Similarly, the antiholomorphic part of the effective action can be found from (4.29) when we restrict the strength $W$ to be constant

$$W = \mathbf{W} = const. \quad (4.35)$$

The antiholomorphic part now reads

$$\delta\Gamma_{antihol} = \frac{1}{16\pi^2}\int d^{12}z du\, \delta V^{++} \star V^{--} \star \ln_\star \frac{\bar{W}}{\Lambda}. \tag{4.36}$$

In contrast to the variation (4.33), the expression (4.36) can not be so easily integrated since there is no antiholomorphic potential written in the antichiral superspace. However, one can readily find the (part of) effective equation of motion corresponding to the variation (4.36)

$$\frac{\delta\Gamma_{antihol}}{\delta V^{++}} = \frac{1}{16\pi^2}(\bar{D}^+)^2\left[\bar{W} \star \ln_\star \frac{\bar{W}}{\Lambda}\right]. \tag{4.37}$$

The variation (4.36) will be integrated in the next section only for some particular choice of background gauge superfields.

# 5 Component structure of low-energy effective action

We are interested in the leading component terms of the (anti)holomorphic effective actions (4.34) and (4.36) in the bosonic sector. Clearly, these effective actions should give the corrections to the corresponding expression (2.11) in the undeformed theory. Therefore we will use the same ansatz (2.10) for the component fields. Here we will follow the works [12, 13], using the same conventions and notations for the component fields and sigma-matrices.

The scalar and vector fields enter the prepotential $V^{++}$ in the Wess-Zumino gauge as follows

$$V^{++}_{WZ} = (\theta^+)^2\bar{\phi}+(\bar{\theta}^+)\phi+(\theta^+\sigma_m\bar{\theta}^+)A_m-2i(\bar{\theta}^+)^2(\theta^+\theta^-)\partial_m A_m-(\bar{\theta}^+)^2(\theta^-\sigma_{mn}\theta^+)F_{mn}. \tag{5.1}$$

The prepotential $V^{--}$ is defined as a solution of zero-curvature equation (3.6). Unfolding the star-product in (3.6), we have

$$\begin{aligned} &D^{++}V^{--} - D^{--}V^{++}_{WZ} + 2I[\partial^\alpha_+ V^{++}_{WZ}\partial_{-\alpha}V^{--} - \partial^\alpha_- V^{++}_{WZ}\partial_{+\alpha}V^{--}] \\ &+\frac{I^3}{2}[\partial^\alpha_-(\partial_+)^2 V^{++}_{WZ}\partial_{+\alpha}(\partial_-)^2V^{--} - \partial^\alpha_+(\partial_-)^2V^{++}_{WZ}\partial_{-\alpha}(\partial_+)^2V^{--}] = 0, \end{aligned} \tag{5.2}$$

where

$$\partial_{+\alpha} = \frac{\partial}{\partial\theta^{+\alpha}}, \qquad \partial_{-\alpha} = \frac{\partial}{\partial\theta^{-\alpha}}. \tag{5.3}$$

One can look for the prepotential $V^{--}$ in the following form

$$\begin{aligned} V^{--} \;=\; & v^{--} + \bar{\theta}^-_{\dot\alpha} v^{-\dot\alpha} + (\bar{\theta}^-)^2 A + (\bar{\theta}^+\bar{\theta}^-)\varphi^{--} \\ &+(\bar{\theta}^+\tilde{\sigma}^{mn}\bar{\theta}^-)\varphi^{--}_{mn} + (\bar{\theta}^-)^2\bar{\theta}^+_{\dot\alpha}\tau^{-\dot\alpha} + (\bar{\theta}^+)^2(\bar{\theta}^-)^2\tau^{--}, \end{aligned} \tag{5.4}$$

where all fields in the rhs of eq. (5.4) depend only on $\theta^+_\alpha$, $\theta^-_\alpha$ variables. The superfields $v^{--}$, $v^{-\dot\alpha}$, $\varphi^{--}$, $A$, $\tau^{-\dot\alpha}$, $\tau^{--}$ should be found from the eq. (5.2). The iterative procedure of solving eq. (5.2) is given in [12]. Following the same steps we find

$$v^{--} = (\theta^-)^2 \frac{\bar\phi}{1+4I\bar\phi}, \tag{5.5}$$

$$v^{-\dot\alpha} = \frac{(\theta^-\sigma_m)^{\dot\alpha} A_m}{1+4I\bar\phi}, \tag{5.6}$$

$$\varphi^{--} = -\frac{2i(\theta^-)^2 \partial_m A_m}{1+4I\bar\phi}, \tag{5.7}$$

$$A = \phi + \frac{4I A_m A_m}{1+4I\bar\phi} + (\theta^+\sigma_{mn}\theta^-)F_{mn}, \tag{5.8}$$

$$\tau^{-\dot\alpha} = \frac{4I(\theta^-\sigma_{mn})^\alpha F_{mn}\sigma_{r\alpha}{}^{\dot\alpha} A_r}{1+4I\bar\phi}, \tag{5.9}$$

$$\tau^{--} = \frac{4I(\theta^-)^2(F_{mn}F_{mn} + F_{mn}\tilde F_{mn})}{1+4I\bar\phi}. \tag{5.10}$$

Now we obtain the component structure of the strength superfields

$$W = -\frac{1}{4}(\bar D^+)^2 V^{--} = \phi + \frac{4IA_mA_m}{1+4I\bar\phi} + (\theta^+\sigma_{mn}\theta^-)F_{mn}, \tag{5.11}$$

$$\bar W = -\frac{1}{4}(D^+)^2 V^{--} = \frac{\bar\phi}{1+4I\bar\phi} + (\bar\theta^+\tilde\sigma_{mn}\bar\theta^-)\frac{F_{mn}}{1+4I\bar\phi}. \tag{5.12}$$

Note that the strength superfields (5.11) and (5.12) are deformed differently. It is clear that in the limit $I \to 0$ these expressions coincide with the undeformed ones (2.9). Introducing the notations

$$\Phi = \phi + \frac{4IA_mA_m}{1+4I\bar\phi}, \qquad \bar\Phi = \frac{\bar\phi}{1+4I\bar\phi}, \qquad \mathbf{F}_{mn} = \frac{F_{mn}}{1+4I\bar\phi}, \tag{5.13}$$

the eqs. (5.11,5.12) can be written in a form similar to eq. (2.9)

$$W = \Phi + (\theta^+\sigma_{mn}\theta^-)F_{mn}, \qquad \bar W = \bar\Phi + (\bar\theta^+\tilde\sigma_{mn}\bar\theta^-)\mathbf{F}_{mn}. \tag{5.14}$$

To find the component structure of the holomorphic potential (4.34,4.36) we have to compute the following quontities

$$W\star W = \Phi^2 + 4I^2(F^2 + F\tilde F) + 2(\theta^+\sigma_{mn}\theta^-)F_{mn}\Phi + (\theta^+)^2(\theta^-)^2(F^2+F\tilde F), \tag{5.15}$$

$$\begin{aligned} \ln_\star \frac{W}{\Lambda} &= \ln\frac{\Phi}{\Lambda} + \frac{1}{4}\ln\left[1 - \frac{8I^2(F^2+F\tilde F)}{\Phi^2}\right] + (\theta^+\sigma_{mn}\theta^-)\frac{F_{mn}}{\Phi}\frac{\operatorname{arcth}\sqrt{\frac{8I^2(F^2+F\tilde F)}{\Phi^2}}}{\sqrt{\frac{8I^2(F^2+F\tilde F)}{\Phi^2}}} \\ &+ \frac{1}{16I^2}(\theta^+)^2(\theta^-)^2 \ln\left[1 - \frac{8I^2(F^2+F\tilde F)}{\Phi^2}\right], \end{aligned} \tag{5.16}$$

$$\bar W\star\bar W = \bar W\bar W = \bar\Phi^2 + 2(\bar\theta^+\tilde\sigma_{mn}\bar\theta^-)F_{mn}\bar\Phi + (\bar\theta^+)^2(\bar\theta^-)^2(F^2+F\tilde F), \tag{5.17}$$

$$\ln_\star\frac{\bar W}{\Lambda} = \ln\frac{\bar W}{\Lambda} = \ln\frac{\bar\Phi}{\Lambda} + (\bar\theta^+\tilde\sigma_{mn}\bar\theta^-)\frac{\mathbf{F}_{mn}}{\bar\Phi} - \frac{1}{2}(\bar\theta^+)^2(\bar\theta^-)^2\frac{\mathbf{F}^2 + \mathbf{F}\tilde{\mathbf{F}}}{\bar\Phi^2}. \tag{5.18}$$

As a result, substituting the expressions (5.15,5.16) into eq. (4.34), we find the component structure of the holomorphic effective action

$$\Gamma_{hol} = -\frac{1}{32\pi^2}\int d^4x(F^2+F\tilde{F})\left[\ln\frac{\Phi}{\Lambda}+\Delta(X(\Phi,F_{mn}))\right], \tag{5.19}$$

where

$$\Delta(X) = \frac{1}{2}(1-X)^2\ln(X-1)+\frac{1}{2}(1+X)^2\ln(1+X)-(1+X^2)\ln X, \tag{5.20}$$

$$X(\Phi,F_{mn}) = \frac{\Phi}{2I\sqrt{2(F^2+F\tilde{F})}}. \tag{5.21}$$

The equation (5.19) shows that the function $\Delta(X)$ is a correction due to non-anticommutativity to the standard holomorphic effective action (2.11). This function has the smooth limit at $I\to 0$

$$\lim_{I\to 0}\Delta(X)=\frac{3}{2}. \tag{5.22}$$

Note that exactly the constant $3/2$ stands in the rhs in eq. (2.11) which was not essential in the undeformed theory, but now this constant is replaced by the function $\Delta(X)$. More detailed studies of the function $\Delta(X)$ are given in the Appendix.

Let us consider the antiholomorphic potential when the strength superfield $\bar{W}$ is defined by the component expression (5.12) with the fields $F_{mn}$ and $\bar{\phi}$ being constant. The equations (5.17,5.18) show that the star-product can be omitted for such an approximation. Therefore the variation (4.36) can be integrated is the same way as in the undeformed theory

$$\begin{aligned}\Gamma_{antihol} &= -\frac{1}{32\pi^2}\int d^4x d^4\bar{\theta}\,\bar{W}^2\ln\frac{\bar{W}}{\Lambda} = -\frac{1}{32\pi^2}\int d^4x(\mathbf{F}^2+\mathbf{F}\tilde{\mathbf{F}})\left(\ln\frac{\bar{\Phi}}{\Lambda}+\frac{3}{2}\right)\\ &= -\frac{1}{32\pi^2}\int d^4x\frac{(F^2+F\tilde{F})}{(1+4I\bar{\phi})^2}\left(\ln\frac{\bar{\phi}}{\Lambda(1+4I\bar{\phi})}+\frac{3}{2}\right).\end{aligned} \tag{5.23}$$

Here the equations (5.17,5.18,5.13) have been used. We see that the non-anticommutativity manifests itself here by a simple rescaling of fields by the factor $1/(1+4I\bar{\phi})$. In the limit $I\to 0$ the expression (5.23) reduces to the standard one for the antiholomorphic potential.

# 6 Deformation of the central charge and the mass

It is well known [20] that the model of hypermultiplet interacting with the external vector superfield possesses non-trivial central charge which is related to the mass of the hypermultiplet via BPS relation. Let us find the deformation of central charge and the mass in the case of non-anticommutative singlet deformation under considerations.

The central charges in the $\mathcal{N}=2$ theories arise effectively from non-vanishing vacuum expectation values of scalar fields [20]

$$\langle\phi\rangle = a, \qquad \langle\bar{\phi}\rangle = \bar{a}. \tag{6.24}$$

The constants $a$, $\bar{a}$ enter the prepotentials (5.1,5.4) as follows

$$\mathbf{V}^{++} = a(\bar{\theta}^+)^2 + \bar{a}(\theta^+)^2, \tag{6.25}$$

$$\mathbf{V}^{--} = a(\bar{\theta}^-)^2 + \frac{\bar{a}}{1+4I\bar{a}}(\theta^-)^2. \tag{6.26}$$

The corresponding strength superfields (5.11,5.12) read now

$$\mathbf{W} = a, \qquad \bar{\mathbf{W}} = \frac{\bar{a}}{1+4I\bar{a}}. \tag{6.27}$$

We see that the strength $\bar{\mathbf{W}}$ is deformed by the non-anticommutativity while $\mathbf{W}$ is not.

Now, using the standard relations $V^-_\alpha = -D^+_\alpha V^{--}$, $\bar{V}^-_{\dot\alpha} = -\bar{D}^+_{\dot\alpha} V^{--}$, we derive the covariant spinor derivatives, corresponding to the prepotentials (6.25,6.26)

$$\begin{array}{ll} \bar{\mathbf{D}}^+_{\dot\alpha} = \bar{D}^+_{\dot\alpha}, & \mathbf{D}^+_\alpha = D^+_\alpha, \\ \bar{\mathbf{D}}^-_{\dot\alpha} = \bar{D}^-_{\dot\alpha} + 2a\bar{\theta}^-_{\dot\alpha}, & \mathbf{D}^-_\alpha = D^-_\alpha - \dfrac{2\bar{a}}{1+4I\bar{a}}\theta^-_\alpha. \end{array} \tag{6.28}$$

The supercharges anticommuting with the derivatives (6.28) are

$$\begin{array}{ll} \bar{\mathbf{Q}}^+_{\dot\alpha} = \bar{Q}^+_{\dot\alpha} - 2a\bar{\theta}^+_{\dot\alpha}, & \mathbf{Q}^+_\alpha = Q^+_\alpha + \dfrac{2\bar{a}}{1+4I\bar{a}}\theta^+_\alpha, \\ \bar{\mathbf{Q}}^-_{\dot\alpha} = \bar{Q}^-_{\dot\alpha}, & \mathbf{Q}^-_\alpha = Q^-_\alpha. \end{array} \tag{6.29}$$

Note that only the supercharge $\mathbf{Q}^+_\alpha$ is deformed by the non-anticommutativity. It is easy to find now the anticommutation relation between the supercharges (6.29)

$$\{\bar{\mathbf{Q}}^+_{\dot\alpha}, \bar{\mathbf{Q}}^-_{\dot\beta}\} = 2\bar{Z}\varepsilon_{\dot\alpha\dot\beta}, \qquad \{\mathbf{Q}^+_\alpha, \mathbf{Q}^-_\beta\} = -2Z\varepsilon_{\alpha\beta}, \tag{6.30}$$

where the central charges $Z$, $\bar{Z}$ are given by

$$\bar{Z} = a, \qquad Z = \frac{\bar{a}}{1+4I\bar{a}}. \tag{6.31}$$

As a result, only the central charge $Z$ is deformed by the non-anticommutativity while $\bar{Z}$ is not.

It is well known that in the presence of central charge the hypermultiplet acquires the BPS mass. To find the mass, let us consider the operator

$$\hat{\Box} = -\frac{1}{2}(D^+)^4(\mathbb{\nabla}^{--})^2 = \Box - m^2, \tag{6.32}$$

where $\mathbb{V}^{--} = D^{--} + \mathbf{V}^{--}$ and

$$m^2 = Z\bar{Z} = \frac{a\bar{a}}{1 + 4I\bar{a}}. \tag{6.33}$$

It is easy to see that $m^2$, given by eq. (6.33), is a mass squared of the hypermultiplet. Indeed, let us consider the hypermultiplet model interacting with the background vector superfield (6.25)

$$\mathbf{S} = \int d\zeta^{(-4)} du\, \breve{q}^{+} \star (D^{++} + \mathbf{V}^{++}) \star q^{+}. \tag{6.34}$$

The model (6.34) is effectively described by the massive propagator

$$\mathbf{G}^{(1,1)}(1|2) = -\frac{1}{\Box - m^2}(D_1^{+})^4 (D_2^{+})^4 \left\{ e_{\star}^{\mathbf{\Omega}(1)} \star e_{\star}^{-\mathbf{\Omega}(2)} \star \frac{\delta^{12}(z_1 - z_2)}{(u_1^{+} u_2^{+})^3} \right\}, \tag{6.35}$$

which can be derived in the same way as the general one (3.11). Here $\mathbf{\Omega}$ is a bridge superfield corresponding to the prepotential $\mathbf{V}^{++}$ (6.25).

Note that the mass squared (6.33) and the central charges (6.31) are deformed in such a way that the BPS relation $m^2 = Z\bar{Z}$ is conserved.

# 7 Conclusions

In this paper we studied the low-energy effective action and renormalizability of $\mathcal{N}$=(1,0) non-anticommutative charged hypermultiplet theory. This model describes the interaction of a hypermultiplet with an Abelian vector superfield under the singlet chiral deformation of supersymmetry. Let us summarize the basic results obtained in the present work.

1. The procedure of perturbative quantum computation of the low-energy effective action for the deformed charged hypermultiplet model is developed within the harmonic superspace approach.

2. Using this procedure, the divergent part of the effective action is calculated and is shown to be proportional to the classical action of $\mathcal{N}$=(1,0) non-anticommutative SYM theory. In this sense the present model is renormalizable.

3. The general structure of the low-energy effective action of this theory is revealed. Away from the undeformed limit, the antiholomorphic piece no longer exists by itself but is incorporated in a *full* $\mathcal{N}$=(1,1) superspace integral.

4. The holomorphic effective action is calculated and remains a *chiral* superspace integral. It is shown to be given by the holomorphic potential which is a star-generalization of the undeformed one. The contribution to (the variation of) the effective action that corresponds to the antiholomorphic potential is also found as an expression written in full $\mathcal{N}$=(1,1) superspace.

5. The component structure of the (anti)holomorphic effective action is studied in the bosonic sector in the constant-fields approximation. It is shown that the holomorphic and antiholomorphic potentials still get deformed differently. In this approximation the deformed holomorphic potential (5.19) acquires the extra terms given by the function (5.20). For the antiholomorphic piece, it is shown that the deformation merely effects a rescaling of component fields by a factor of $(1 + 4I\bar{\phi})^{-1}$, where $\bar{\phi}$ is one of the two scalar fields of the vector multiplet.

6. The deformation of the mass and the central charge is found. It is shown that the mass-squared as well as the central charge are rescaled by the same factor $(1+4I\bar{\phi})^{-1}$, preserving the relation between them.

In the light of the present results, it would be rewarding to solve the following problems concerning the quantum aspects of non-anticommutative theories with extended supersymmetry. First, it is tempting to determine for the hypermultiplet model the deformation of the next-to-leading terms in the effective action, which are necessarily non-holomorphic. Also, one should develop the non-Abelian generalization. Next, it is important to perform an analogous investigation for the pure SYM theory, since in the undeformed case this model (the Seiberg-Witten theory) plays an important role in modern theoretical physics. Finally, it would be interesting to extend the quantum studies of non-anticommutative theories to the case of non-singlet (i.e. more general) deformations of supersymmetry, as considered particularly in [10] on the classical level.


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# A Appendix

Let us give some more comments about the function $\Delta(X)$ given by eq. (5.20).

**i.** Due to the presence of log function, the expression (5.20) is well-defined if $X > 1$, or

$$\Phi > 2I\sqrt{2(F^2 + F\tilde{F})}. \tag{A.1}$$

The equation (A.1) means that the vacuum values for the scalar field $\phi$ are bounded below if $I \neq 0$.

**ii.** In the region (A.1) the function $3/2 - \Delta(X)$ is monotone decreasing. It is plotted in the Fig. 1a.

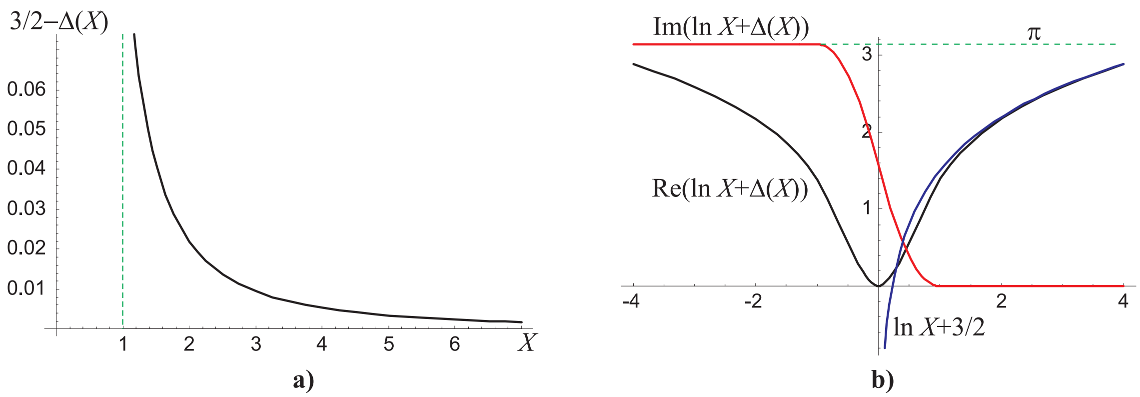


Fig. 1.

**iii.** If one relaxes the condition (5.22), the function $\Delta(X)$, as well as the holomorphic effective action, acquires an imaginary part. The expressions $\mathrm{Re}(\ln X + \Delta(X))$, $\mathrm{Im}(\ln X + \Delta(X))$ are plotted in Fig. 1b in comparison with the function $\ln X + 3/2$ which is responsible for the bosonic part of the holomorphic effective action (2.11) in the undeformed case. It is interesting to note that the function $\ln X + \Delta(X)$ has no logarithmic singularity at the origin $X = 0$ when $I \neq 0$.

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